Shear Capacity of RC Membrane Elements Subjected to Pure In-Plane Shear Stresses

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ABSTRACT

This study evaluates the shear capacity of reinforced concrete (RC) membrane elements subjected to pure in-plane shear stresses at different levels of loading. These elements are shown to fail in four different modes. A model is proposed for the shear capacity of these membrane elements that can be computed by direct simple expressions without the need to any iterative process as is usually the case in existing models. Failure in these membrane elements is generally preceded by the formation of two or more major critical cracks assumed herein to propagate in a direction normal to the principal tensile stresses in the concrete. At each stage of loading, the major critical cracks are shown to open at an optimum angle that corresponds to the least shearing resistance to external loading. Major cracks as well as ultimate shear capacity of RC membrane elements are shown to occur when the contribution of the x-reinforcement to resist shear stresses is equal to that of the y-reinforcement. Experimental results of tests on RC membrane elements subjected to pure in-plane shear stresses obtained from literature are used to validate and compare the proposed model with existing models. Examples are provided for the computations of shear capacity by the proposed model for the different modes of failure.

KEYWORDS: Reinforced concrete, Membrane element, Yield, Shear capacity, Critical cracks, Modes of failure.

INTRODUCTION

Many large reinforced concrete (RC) structures, such as shear walls, containment vessels, thin-wall structures,… etc., can be modeled as a series of two-dimensional membrane elements subjected to in-plane stresses. The problem is thus reduced to predicting the behavior of a single RC membrane element (Vecchio and Collins, 1986; Rahal, 2002). Reinforced concrete membrane elements subjected to in-plane stresses may fail in four different modes, as confirmed in this paper, depending on the amounts of transverse and longitudinal reinforcements and concrete strength as well.

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Recent years have witnessed important advances in the analysis of RC structures. These advances involve predicting the behavior of RC members, including their strength and load-deformation response, which has been the subject of intensive research for more than a century. Although many theories and design procedures or models for structural concrete subjected to shear have been proposed (Vecchio and Collins, 1986; Bažant and Kazemi, 1991; CSA, 1994; Belarbi and Hsu, 1994, 1995; ; Zararis, 1995, 1996; Hsu, 1991, 1996, 1998; Hsu and Zhang, 1996, 1997; Kaufmann and Marti, 1998; Carbone et al., 2001; Bentz et al., 2006; Kazemi and Broujerdian, 2006; Rahal, 2008; Broujerdian and Kazemi, 2010; Miguel et al., 2013), no universally accepted rational method of shear design has evolved (Broujerdian and Kazemi, 2010) and the determination of shear strength of RC beams remains open to discussion. The shear strengths predicted by the different current design codes for a particular beam section can vary by factors of more than 2 (Bentz et al., 2006). The complexity of the shear problem can mainly be attributed to the complex nature of diagonal tension cracks and their secondary effects. Reinforced concrete beams transmit shear in a relatively complex manner (ASCE-ACI-445-99). Some of the related problems are initiation of new cracks, closing or propagation of preexisting cracks, variation of stresses along the reinforcing bars and variations of tensile and compressive stresses in concrete between cracks (Broujerdian and Kazemi, 2010). Most of the above-mentioned models involve predicting the strength and load-deformation response of RC membrane elements, require constitutive laws for both steel and concrete (in tension and compression) and involve solving a set of equations with an iterative process.

In this paper, a model is proposed to assess the behavior of orthogonally reinforced concrete membrane elements under pure in-plane shear stresses incorporating analytical relations and formulae governing the behavior of RC membrane elements at different levels of loading. The shear capacity of these membrane elements is directly obtained by simple straightforward expressions without using an iterative process. The proposed model is developed based on the fact that cracks in RC membrane elements open in a direction perpendicular to the principal tensile stresses in concrete, as observed from experiments (Leonhardt and Schelling, 1974). At each stage of loading, these major critical cracks are shown to open at an angle that assumes least shearing resistance to external loading. The shear strengths of RC membrane elements predicted by the proposed model were found to be quite similar to the corresponding experimental values as well as to the predictions of the modified compression field theory (MCFT) of Vecchio and Collins (1986).

Formation of the First Major Critical Cracks

Failure in RC membrane elements is generally preceded by the formation of two or more major critical cracks depending on the level of loading. This section is devoted to develop an analytical relationship for the inclination of the first major critical cracks at the instant these first cracks open based on the concept of optimization.

The major critical cracks open in a direction normal to the principal tensile stresses in concrete when these principal tensile stresses reach the tensile strength of concrete. Figure 1 shows the orientations of the principal tensile and compressive stresses in concrete prior to cracking by a Mohr circle, where ϕ is the angle between the principal plane on which the principal tensile stresses act and the y-axis. The coordinate axes x and y are assumed to coincide with the direction of the longitudinal and transverse reinforcements, respectively.

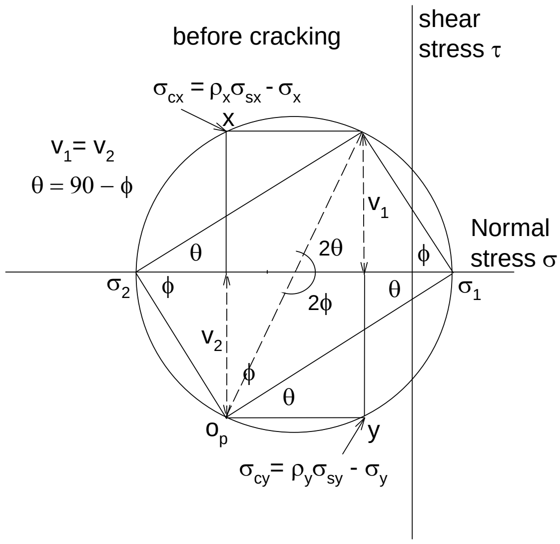

Figure (1): Mohr circle of concrete stresses prior to formation of first major critical cracks

At the instant the first major critical cracks open, the concrete tensile stresses are lost due the opening of cracks. The concrete tensile stresses at these cracks are reduced to zero after cracks open (Belarbi and Hsu, 1995). The angle $\phi = \phi_{cr}$ then represents the direction of first major critical cracks with the y-axis.

Upon the formation of these first major critical cracks, strains ε_{cr} orthogonal to the cracks will cause the x and y reinforcements to undergo both normal and shear strains. Assuming that the stresses in both x and y reinforcements are in the elastic range ($\sigma_{sx} = E_s \varepsilon_{sx} < f_{sy-x}$ and $\sigma_{sy} = E_s \varepsilon_{sy} < f_{sy-y}$: where E_s is the modulus of elasticity and f_{sy-x} and f_{sy-y} are respectively the yield strengths of the x and y reinforcements), the normal stresses in the x and y reinforcements can be related (Zararis, 1996) by the following expression:

$$\sigma_{sx} = \sigma_{sy} \cot^2 \phi_{cr} . \qquad (1)$$

Furthermore, the shear stresses τ_{sxy} and τ_{syx} in the x and y reinforcements in the elastic range ($\tau = G\gamma$ where $G = E/2(1+v)$) can be expressed as a function of the normal stresses in the x and y reinforcements (σ_{sx} and σ_{sy}), respectively, as follows:

$$\tau_{sxy} = G\gamma_{sxy} \cong 0.40 \sigma_{sx} \tan \phi_{cr} ; \qquad (2)$$

$$\tau_{syx} = G\gamma_{syx} \cong 0.40 \sigma_{sy} \cot \phi_{cr} . \qquad (3)$$

The forces that act on an RC membrane element at the formation of the first major critical cracks that maintain the equilibrium of the membrane element are shown in Fig. 2, where the steel normal and shear stresses are still in the elastic range. The summation of forces normal to crack opening yields the following expression for the shear stresses:

$$v = 0.50 \begin{pmatrix} \rho_x \sigma_{sx} \cot \phi_{cr} + \rho_y \sigma_{sy} \tan \phi_{cr} \\ + \rho_x \tau_{sxy} + \rho_y \tau_{syx} \end{pmatrix} ; \qquad (4)$$

where $v = \tau_{xy}$ is the applied shear stress; ρ_x and ρ_y are the steel ratios in the x and y directions, respectively.

Hence, the shear stress of a given RC membrane element (ρ_x, ρ_y, f_{sx-y}, f_{sy-y}, f_c') can be expressed in terms of σ_{sx}, σ_{sy} and ϕ_{cr} {i.e., $v = f(\sigma_{sx}, \sigma_{sy}, \phi_{cr})$} by substituting Eq. 2 and 3 into Eq. 4 as follows:

$$v = 0.50 \begin{pmatrix} (\rho_x \sigma_{sx} + 0.40 \rho_y \sigma_{sy}) \cot \phi_{cr} \\ + (\rho_y \sigma_{sy} + 0.40 \rho_x \sigma_{sx}) \tan \phi_{cr} \end{pmatrix} . \qquad (5)$$

Equation 5 can be represented by a Mohr circle of stresses as shown in Fig. 3.

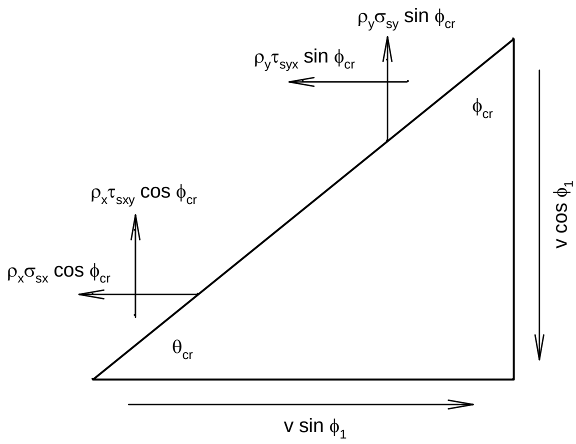

Figure (2): Forces on an RC membrane element after formation of first major critical crack (steel is still in elastic range)

The major critical cracks are assumed to open when the resistance of RC membrane element to external loading is minimum. Accordingly, the angle of inclination of the first major critical cracks can be obtained by minimizing the shear stress v with respect to ϕ_{cr} (i.e., $\partial v / \partial \phi_{cr} = 0$), which is the only possible real solution in this case, as follows:

$$-(\rho_x \sigma_{sx} + 0.40 \rho_y \sigma_{sy}) \cot \phi_{cr} + (\rho_y \sigma_{sy} + 0.40 \rho_x \sigma_{sx}) \tan \phi_{cr} = 0. \qquad (6)$$

Equation 6 can also be obtained from the Mohr circle of stresses depicted in Fig. 3 such that $v_1 = v_2$.

Substituting Eq. 1 into Eq. 6, where $\sigma_{sx} \neq 0$, the following equation is obtained for the angle ϕ_{cr} of the inclination of the first major critical cracks with respect to the y axis:

$$\rho_y \tan^4 \phi_{cr} + 0.40\left(\rho_x - \rho_y\right)\tan^2 \phi_{cr} - \rho_x = 0 \,. \qquad (7)$$

The orientation of the first major critical cracks is analytically proved in this study to provide least shear resistance to external loading at the instant these cracks open using the concept of optimization as explicitly stated by Zararis (1996).

The angle of inclination of the first major critical cracks can explicitly be expressed as follows:

$$\tan\phi_{cr} = \sqrt{\sqrt{0.04\left(\frac{\rho_x}{\rho_y} - 1\right)^2 + \frac{\rho_x}{\rho_y}} - 0.20\left(\frac{\rho_x}{\rho_y} - 1\right)} \qquad (8)$$

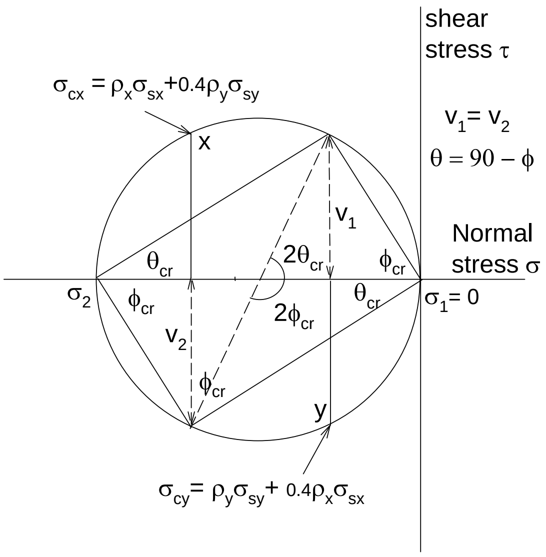

Figure (3): Representation of stresses after formation of first major critical cracks by Mohr circle

The angle of inclination of the first major critical cracks, for a given steel ratio in the x direction ($\rho_x = 0.0179$), decreases with the steel ratio in the y direction (Fig. 4a). On the other hand, the angle of inclination of the first major critical cracks, for a given steel ratio in the y direction ($\rho_y = 0.0089$), increases with the steel ratio in the x direction (Fig. 4b). It should be pointed out that Eq. 8 yields the same results as obtained by the expression derived by Zararis (1996) for the case of pure in-plane shear stresses supporting the validity of the proposed approach in obtaining the shear capacity by the concept of optimization.

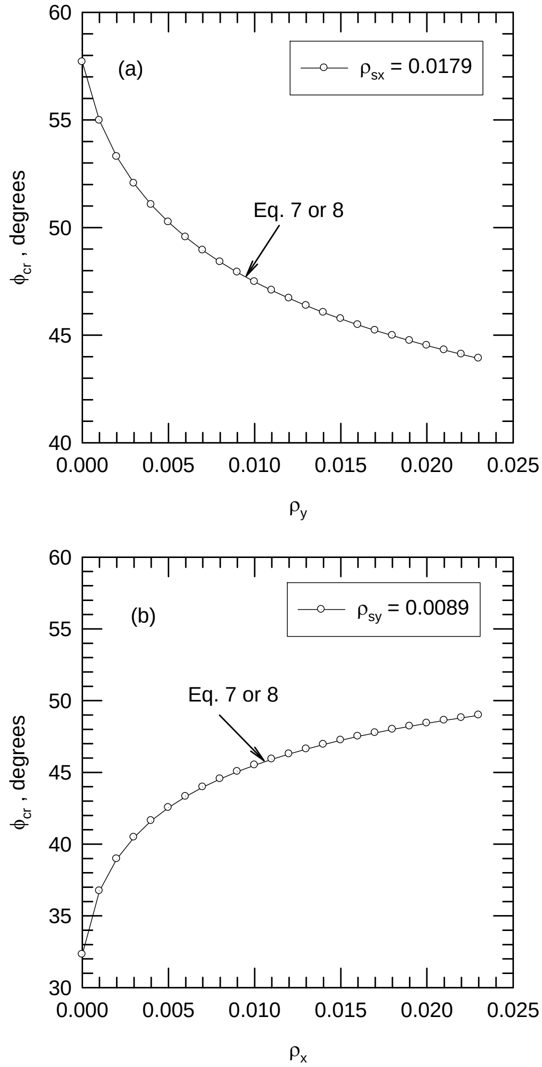

Figure (4): (a) Effect of y-reinforcement on the inclination of the first major critical cracks for a given x reinforcement, (b): Effect of x-reinforcement on the inclination of the first major critical cracks for a given y reinforcement

Failure Modes Following the Formation of the First Major Critical Cracks

As the loading continues to increase following the formation of the first major critical cracks, four modes of failure can be recognized. In the first mode (Mode I), both x and y reinforcements yield prior to concrete crushing (occurs in entirely under-reinforced membrane elements). In the second mode (Mode II), concrete crushes prior to the yielding of any reinforcement (occurs in entirely over-reinforced membrane elements). In the third mode (Mode III), the y reinforcement yields before the yielding of the x reinforcement and prior to concrete crushing. In the fourth mode (Mode IV), the x reinforcement yields before the yielding of the y reinforcement and prior to concrete crushing. Modes III and IV occur in partially under-reinforced membrane elements either in x or y. The analysis of Modes III and IV is the same as long as the weaker reinforcement can readily be identified. Balanced steel ratios in the x and y directions will be defined later in this study in order to determine the modes of failure of RC membrane elements. All these modes are generally recognized by the formation of more than one major critical crack.

At the Yield of Both X and Y Reinforcements (Failure Mode I)

In this mode (Mode I), following the formation of the first major critical cracks at the angle ϕ_{cr} (Eq. 8) as the load continues to increase, the stresses in the x and y reinforcements increase up to their yield strengths and new cracks (second major critical cracks) are formed in another direction, assumed at an angle ϕ_I, that provides least shear resistance at this stage of loading as shown below.

At the simultaneous yielding of both x and y reinforcements ($\sigma_{sx} = f_{sy-x}$ and $\sigma_{sy} = f_{sy-y}$), prior to concrete crushing, the shear capacity can be written in the following form:

$$v_{u-I} = 0.5\left(\rho_x f_{sy-x} \cot\phi_I + \rho_y f_{sy-y} \tan\phi_I\right); \qquad (9)$$

where the steel shear stresses are lost due to the yielding of reinforcements. Shear stresses in the steel reinforcements may be lost due to a slip between the crack faces or yielding of steel (Zararis, 1995).

The angle of inclination ϕ_I of the second major critical cracks (for mode I at simultaneous yield of both reinforcements) can be obtained by minimizing the shear stress v_{u-I} with respect to the angle ϕ_I (i.e., $\partial v_{u-I} / \partial\phi_I = 0$), which is the only possible real solution in this case, as follows:

$$\tan\phi_I = \sqrt{\frac{\rho_x f_{sy-x}}{\rho_y f_{sy-y}}}\,. \qquad (10)$$

Hence, the shear capacity at the simultaneous yielding of both x and y reinforcements can alternatively be expressed using Eq. 9 and 10 with the following well-known form:

$$v_{u-I} = \sqrt{\rho_x f_{sy-x} \rho_y f_{sy-y}}\,. \qquad (11)$$

Belarbi and Hsu (1995) and Rahal (2008) reached the same results as given in Eq. 11 for the case of simultaneous yielding of both x and y reinforcements confirming the validity of the proposed approach in obtaining the shear capacity by the concept of optimization.

Equation 10 can be expressed as:

$$\rho_x f_{sy-x} \cot\phi_I = \rho_y f_{sy-y} \tan\phi_I\,. \qquad (12)$$

Therefore, the shear capacity may be expressed either in terms of x or y reinforcement as follows:

$$v_{u-I} = \rho_x f_{sy-x} \cot\phi_I = \rho_y f_{sy-y} \tan\phi_I\,. \qquad (13)$$

Equations 9 and 12 imply that the failure of orthogonally RC membrane elements occurs when the contribution of the x-reinforcement to resist shear stresses is equal to that of the y-reinforcement.

Balanced Steel Ratios and Modes of Failure

In order to determine the modes of failure of RC membrane elements, a balanced steel ratio must be defined first at which failure of concrete (concrete crushing) takes place simultaneously with the yielding of both x and y reinforcements.

Experimental results for RC membrane elements that failed in different failure modes are depicted in Fig. 5. These results show that there is a clear boundary between those membrane elements in which failure was recognized by concrete crushing prior to the yielding of any reinforcement and those membrane elements in which failure was recognized by the yielding of one or both reinforcements prior to concrete crushing. In order to define the boundary, the following simplifications are considered.

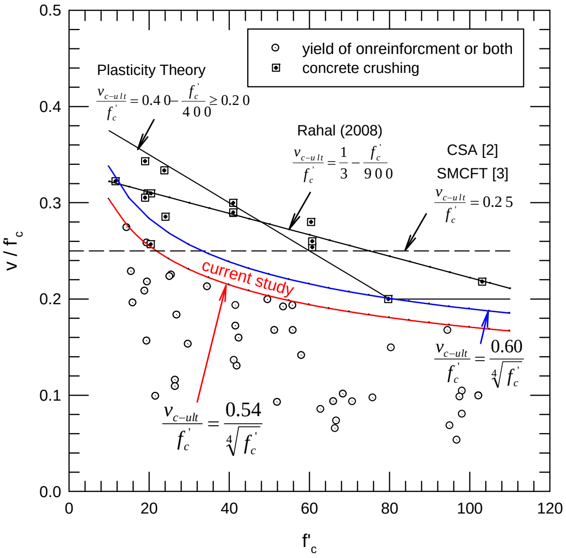

Figure (5): Over-reinforcement limit (ORL) for a wide range of concrete compressive strength that can be used to identify the failure modes

At concrete crushing, the shear stresses of RC membrane elements may be expressed in terms of the principal tensile and compressive stresses (σ_t and σ_c) in the uncracked concrete between existing first major critical cracks by the following simplified form:

$$v_c = \sqrt{\sigma_t \sigma_c} \,. \tag{14}$$

These principal tensile and compressive stresses (σ_t and σ_c) can respectively be expressed at concrete crushing as follows:

$$\sigma_t = k_1 \sqrt{f_c'} \leq f_{ct}' ; \tag{15}$$

$$\sigma_c = k_2 f_c' \leq f_c' ; \tag{16}$$

where f_c' and f_{ct}' are the compressive and tensile strengths of concrete, respectively. Therefore, a limiting shear stress for those membrane elements that failed by concrete crushing may be expressed as follows:

$$v_c = k\left(f_c'\right)^{0.75} ; \tag{17}$$

where $k = \sqrt{k_1 k_2}$. Hence, an over-reinforcement limit (ORL) may be defined at concrete crushing as follows:

$$ORL = \frac{v_{c-ult}}{f_c'} = \frac{k}{\sqrt[4]{f_c'}} \,. \tag{18}$$

Experimental results (Fig. 5) show that the k value can reasonably be taken as 0.57 to be the limit between those RC membrane elements that failed by yielding of one or both reinforcements (circles) prior to concrete crushing and those RC membrane elements that failed by concrete crushing (squares) prior to yielding of any reinforcement. Indeed, $k_b = 0.57$ was taken as the average value between the limit of RC membrane elements that failed by yielding of one or both reinforcements ($k_{yy} = 0.54$) and the limit of RC membrane elements that failed by concrete crushing ($k_{cc} = 0.60$) for the experimental results of Fig. 5. Therefore, the over-reinforcement limit ORL_b for the determination of modes of failure may be proposed as follows:

$$ORL_b = \frac{v_{c-ult}}{f_c'} = \frac{k_b}{\sqrt[4]{f_c'}} = \frac{0.57}{\sqrt[4]{f_c'}} \,. \tag{19}$$

Figure 5 also shows a comparison of the over-reinforcement limit (ORL_b) proposed in this study (Eq. 19) to three expressions usually used as limits to the ultimate shear capacity that can be carried by concrete. These three expressions include that suggested by Rahal (2008), that used in the general procedure of CSA (1994) as well as in the SMCFT (Bentz et al., 2006) and that used based on the plasticity theory (Braestrup, 1974).

If both x and y reinforcements are assumed to yield simultaneously prior to concrete failure, the ultimate shear strength (Eq. 11) can be given by the following normalized expression:

$$\frac{v_{u-I}}{f_c'} = \sqrt{\frac{\rho_x f_{sy-x}}{f_c'} \frac{\rho_y f_{sy-y}}{f_c'}} . \quad (20)$$

Since both x and y reinforcements yield simultaneously prior to concrete failure, the ultimate shear capacity given in Eq. 20 must be less than that of concrete failure (Eq. 19). Therefore,

$$\sqrt{\frac{\rho_x f_{sy-x}}{f_c'} \frac{\rho_y f_{sy-y}}{f_c'}} \le ORL_b . \quad (21)$$

To satisfy this inequality, the following two criteria must be met:

$$\frac{\rho_x f_{sy-x}}{f_c'} \le ORL_b ; \quad (22)$$

$$\frac{\rho_y f_{sy-y}}{f_c'} \le ORL_b . \quad (23)$$

Hence, balanced steel ratios in the x and y directions, which can be used to identify modes of failure, may respectively be defined at concrete crushing with simultaneous yielding of both x and y reinforcements as follows:

$$\rho_{x-b} = \frac{0.57 f_c'^{0.75}}{f_{sy-x}} ; \quad (24)$$

$$\rho_{y-b} = \frac{0.57 f_c'^{0.75}}{f_{sy-y}} . \quad (25)$$

Four modes of failure can be identified from the combinations of these two equations (Eq. 24 and 25) as illustrated in Fig. 6. A comparison of these two criteria with those of Pang and Hsu (1995) is also included in this figure. The first mode (**mode I**): if $\rho_x \le \rho_{x-b}$ and $\rho_y \le \rho_{y-b}$, then both x and y reinforcements will yield (entirely under-reinforced membrane elements); the second mode (**mode II**): if $\rho_x > \rho_{x-b}$ and $\rho_y > \rho_{y-b}$, then both reinforcements do not yield and failure is recognized by concrete crushing (entirely over-reinforced membrane elements); the third mode (**mode III**): if $\rho_x > \rho_{x-b}$ and $\rho_y \le \rho_{y-b}$, then only the y reinforcement will yield (under-reinforced in y direction); the fourth mode (**mode IV**): if $\rho_x \le \rho_{x-b}$ and $\rho_y > \rho_{y-b}$, then only the x reinforcement will yield (under-reinforced in x direction). Table 1 shows that the two proposed criteria (Eqs. 24 and 25) can reliably be used to predict the modes of failure of orthogonally RC membrane elements.

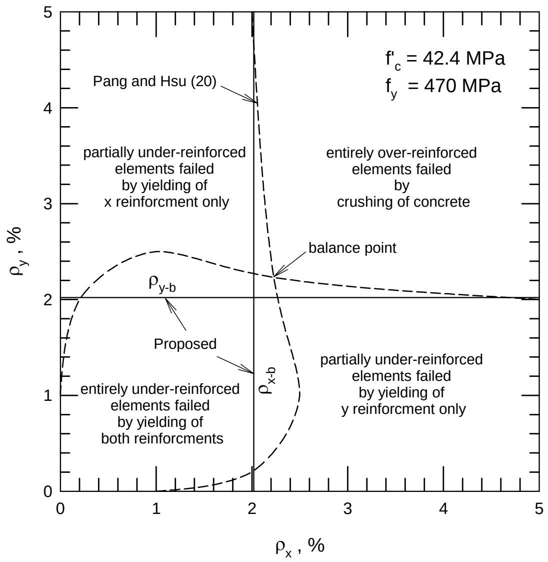

Figure (6): Four failure modes predicted by the two proposed criteria for RC membrane elements loaded in pure shear compared to those of Pang and Hsu (1995)

Verification of Failure Mode I

In the first mode (Mode I), in which both x and y reinforcements yield before concrete failure, ultimate shear strength may be obtained by Eq. 11. Experimental evidence shows that ultimate shear strength of the RC membrane elements in which failure was recognized by simultaneous yielding of both x and y reinforcements prior to concrete crushing can reliably be predicted by Eq. 11 as can be seen in Table 2. The mean and standard deviation of the ratio of experimental to predicted shear strength values for failure mode I are 1.00 and 0.09, respectively.

Table 1. Modes of failure for RC membrane elements subjected to pure in-plane shear stress

	Test	Concrete	Reinforcement						Mode of failure	
Panel	v_{ue} MPa	$f_c^{'}$	ρ_y	f_{sy-y} MPa	ρ_x	f_{sy-x} MPa	ρ_{sy-b}	ρ_{sx-b}	Experimental	Proposed
PV4	2.89	26.6	0.0106	242	0.0106	242	0.0276	0.0276	X,Y	X,Y
PV6	4.55	29.8	0.0179	266	0.0179	266	0.0274	0.0274	X,Y	X,Y
PV9	>3.74	11.6	0.0179	455	0.0179	455	0.0079	0.0079	C	C
PV10	3.97	14.5	0.0100	276	0.0179	276	0.0154	0.0154	Y	Y
PV11	3.56	15.6	0.0131	235	0.0179	235	0.0191	0.0191	X,Y	X,Y
PV12	3.13	16.0	0.0045	269	0.0179	469	0.0169	0.0097	Y	Y
PV14	>5.24	20.4	0.0179	455	0.0179	455	0.0120	0.0120	C	C
PV16	2.14	21.7	0.0074	255	0.0074	255	0.0225	0.0225	X,Y	X,Y
PV18	>3.04	19.5	0.0032	412	0.0179	431	0.0128	0.0123	Y	Y
PV19	3.95	19.0	0.0071	299	0.0179	458	0.0174	0.0113	Y	Y
PV20	4.26	19.6	0.0089	297	0.0179	460	0.0179	0.0116	Y	Y
PV21	5.03	19.5	0.0130	302	0.0179	458	0.0175	0.0116	Y	Y
PV22	6.07	19.6	0.0152	420	0.0179	458	0.0126	0.0116	C	C
PV27	6.35	20.5	0.0179	442	0.0179	442	0.0124	0.0124	C	C
PHS2	6.66	66.1	0.0041	521	0.0323	606	0.0253	0.0218	Y	Y
PHS3	8.19	58.1	0.0082	521	0.0323	606	0.0230	0.0198	Y	Y
PHS8	10.82	55.9	0.0124	521	0.0323	606	0.0223	0.0192	Y	Y
SE1	6.76	42.5	0.0293	492	0.0098	478	0.0193	0.0199	Y	Y
VA4	20.90	103.1	0.0524	470	0.0524	470	0.0393	0.0393	C	C
S-21	6.52	19.0	0.0428	378	0.0428	378	0.0138	0.0138		C
S-34	7.35	34.6	0.0191	418	0.0191	418	0.0194	0.0194		X,Y
S-43	11.87	41.0	0.0428	409	0.0428	409	0.0226	0.0226		C
S-61	15.40	60.7	0.0428	409	0.0428	409	0.0303	0.0303		C
S-81	16.19	79.7	0.0428	409	0.0428	409	0.0371	0.0371		C
Y = yielding of y reinforcement; X, Y = yielding of both x and y reinforcements; C = concrete crushing.										
Experimental results for PV panels are obtained from Vecchio and Collins (1986); VA4 panel are obtained from Zhang and Hsu (1998); PHS panels are obtained from Vecchio et al. (1994); SE1 panels are obtained from Kirschner (1986); S panels are obtained from Yamaguchi et al. (1988).										

Verification of Failure Mode II

For the second mode (Mode II: entirely over-reinforced membrane elements in both x and y directions), in which crushing of concrete occurs prior to the yielding of any reinforcement, the shear strength of RC membrane elements may be estimated by the following formula:

$$v_{u-II} = ORL \ f_c^{'} = k\left(f_c^{'}\right)^{0.75} . \qquad (26)$$

Table 2. Capacity of RC membrane elements subjected to pure in-plane shear stresses

A: Failure Mode I: both x and y reinforcements yield prior to concrete crushing.										
Panel	**Concrete**	**Transverse steel**		**Longitudinal steel**		**Shear capacity**			**Proposed**	**MCFT**
	f_c' **MPa**	ρ_y	f_{sy-y} **MPa**	ρ_x	f_{sy-x} **MPa**	**Experimental** v_{ue} **MPa**	**Proposed** v_{up} **MPa**	**MCFT*** v_{uM} **MPa**	$\frac{v_{ue}}{v_{up}}$	$\frac{v_{ue}}{v_{uM}}$
PV3	26.6	0.0048	662	0.0048	662	3.07	3.17	3.30	0.97	0.93
PV4	26.6	0.0106	242	0.0106	242	2.89	2.57	2.57	1.12	1.12
PV6	29.8	0.0179	266	0.0179	266	4.55	4.76	4.75	0.96	0.96
PV11	15.6	0.0131	235	0.0179	235	3.56	3.85	3.59	0.92	0.99
PV16	21.7	0.0074	255	0.0074	255	2.14	1.89	1.90	1.13	1.13
S-34	34.6	0.0191	418	0.0191	418	7.35	7.98	8.07	0.92	0.91
								Mean	1.00	1.01
								Standard deviation	0.09	0.09
B: Failure Mode II: concrete crushing without the yield of any reinforcement.										
PV9	11.6	0.0179	455	0.0179	455	>3.74	3.77	3.70	0.99	1.01
PV14	20.4	0.0179	455	0.0179	455	>5.24	5.76	-----	0.91	-----
PV22	19.6	0.0152	420	0.0179	458	6.07	5.59	6.14	1.08	0.99
PV27	20.5	0.0179	442	0.0179	442	6.35	5.79	6.36	1.10	1.00
S-21	19.0	0.0428	378	0.0428	378	6.52	5.58	7.33	1.17	0.89
S-43	41.0	0.0428	409	0.0428	409	11.87	9.72	13.04	1.20	0.91
S-61	60.7	0.0428	409	0.0428	409	15.40	13.05	17.11	1.18	0.90
VA4	103.1	0.0524	470	0.0524	470	22.48	19.41	22.70	1.16	0.99
								Mean	1.10	0.95
								Standard deviation	0.10	0.05
C: Failure Mode III: concrete crushing with the yield of the weaker reinforcement.										
PV10	14.5	0.0100	276	0.0179	276	3.97	4.05	3.58	0.98	1.11
PV12	16.0	0.0045	269	0.0179	469	3.13	2.54	2.47	1.23	1.27
PV18	19.5	0.0032	412	0.0179	431	>3.04	2.83	2.82	1.07	1.08
PV19	19.0	0.0071	299	0.0179	458	3.95	3.72	4.16	1.06	0.95
PV20	19.6	0.0089	297	0.0179	460	4.26	4.28	4.51	1.00	0.94
PV21	19.5	0.0130	302	0.0179	458	5.03	5.45	4.88	0.92	1.03
PHS2	66.1	0.0041	521	0.0323	606	6.66	5.39	6.07	1.24	1.10
PHS3	58.1	0.0082	521	0.0323	606	8.19	7.74	8.18	1.06	1.00
PHS8	55.9	0.0124	521	0.0323	606	10.82	9.83	9.90	1.10	1.09
SE1	42.5	0.0098	492	0.0293	478	6.76	7.63	7.04	0.89	0.96
								Mean	1.05	1.05
								Standard deviation	0.11	0.10
D: all failure modes combined.										
								Mean	1.09	1.04
								Standard deviation	0.11	0.11
*Results for MCFT are obtained from Bentz et al. (2006).										

The value of k depends on the level of stresses in the non-yielded reinforcements compared to $k_b = 0.57$ at the balanced conditions. However, for simplicity, the k value is taken constant and is obtained empirically by using the experimental results of Fig. 5, in which crushing of concrete occurs prior to the yielding of any reinforcement. These experimental results suggest that the k value may conservatively be taken as 0.60 for those elements that failed only by crushing of concrete prior to yield of any reinforcement. Therefore, the shear strength of RC membrane elements for mode II can be expressed as follows:

$$v_{u-II} = ORL\ f_c^{'} = 0.60\left(f_c^{'}\right)^{0.75}. \qquad (27)$$

The adoption of $k = 0.60$ partly accounts for the contribution of portion of the non-yielded reinforcements (in excess of the balanced values: $k_b = 0.57$). The ratio of experimental to predicted shear strength values for failure mode II only (Table 2) ranges between 0.91 and 1.20 with a mean and a standard deviation for this ratio of 1.10 and 0.10, respectively. These results show that Eq. 27 is conservative by only 10% compared to experimental data.

Failure Modes III and IV

Modes III and IV can be dealt with in the same way as long as the weaker reinforcement is identified first (y reinforcement in this case). Mode III (or IV) is divided into three stages.

A) First Stage: Formation of the First Major Critical Cracks

In this stage, first major critical cracks are initially formed in a certain direction. Earlier in this study, it was shown that the inclination of these first major critical cracks open at minimum shear resistance to external loads (Eq. 8). For a given RC membrane element (ρ_x , ρ_y , f_{sy-x} , f_{sy-y} and $f_c^{'}$), the angle ϕ_{cr} of the first major critical cracks is constant. However, the stresses at the location of these first cracks change during loading.

B) Second Stage: At the Yield of Transverse (Weaker) Reinforcement

At this stage, it is assumed that the transverse (weaker) reinforcement is oriented along the y axis and that the normal stress σ_{sy} in the y reinforcement at the location of first major critical cracks reaches the yield strength, f_{sy-y}. Hence, the shear stresses at the location of these first cracks at the instant the normal stress σ_{sy} reaches the yield strength f_{sy-y} can be given as follows:

$$v_{u-y} = 0.50\left(\rho_x \sigma_{sx1} \cot\phi_{cr} + \rho_y f_{sy-y} \tan\phi_{cr}\right). \qquad (28)$$

The normal stress σ_{sx1} in the x reinforcement at the location of the first major critical cracks at the instant the y reinforcement yields can be given (Eq. 7: where $\sigma_{sy} = f_{sy-y}$) as follows:

$$\sigma_{sx1} = \frac{f_{sy-y}}{\tan^2\phi_{cr}} < f_{sy-x}. \qquad (29)$$

Therefore, the shear capacity of an RC membrane element just at the yield of the transverse y reinforcement may be expressed as follows:

$$v_{u-y} = \frac{f_{sy-y}}{2}\left(\rho_x \cot^3\phi_{cr} + \rho_y \tan\phi_{cr}\right). \qquad (30)$$

Following the yield of the y reinforcement, this y reinforcement cannot carry any additional loads; therefore, additional loads must be carried by the stronger x reinforcement and by concrete as well. However, as loading continues, new cracks open at a different inclination.

C) Third Stage: At the Formation of the Second Major Critical Cracks

In the third stage of mode III (or IV), prior to the yield of the stronger x reinforcement, second major critical cracks are formed in another direction. It is assumed in this study that the ultimate shear strength occurs just at the formation of these second major critical cracks, because afterwards the membrane elements become unstable and their faces slip rapidly and strongly causing a local crushing of concrete (Vecchio and Collins, 1982; Zararis, 1996).

Following the yield of the transverse y reinforcement, but prior to the yield of the x reinforcement, the shear stress at the second major critical cracks can be expressed as follows:

$$v_{u-III} = 0.50 \begin{pmatrix} \rho_x \sigma_{sx2-3} \cot\phi_{III} \\ + \rho_y f_{sy-y} \tan\phi_{III} \end{pmatrix}; \quad (31)$$

where σ_{sx2-3} is the normal stresses at the location of the second major critical cracks in the x-reinforcement at failure of the membrane element under Mode III (or Mode IV). The shear stress in the x reinforcement is assumed to be lost due the rapid slip described above.

The inclination of the second major critical cracks may be obtained by optimizing the shear capacity (Eq. 31) with respect to ϕ_{III} (i.e., $\partial v_{u-III} / \partial \phi_{III} = 0$) as follows:

$$\tan^2 \phi_{III} = \frac{\rho_x \sigma_{sx2-3}}{\rho_y f_{sy-y}} = \psi_{sxIII} . \quad (32)$$

The inclination of the second major critical cracks provides a minimum shear resistance at this stage of loading. However, the stress σ_{sx2-3} and angle ϕ_{III} are not known yet.

Rearranging Eq. 32 yields the following expression:

$$\rho_x \sigma_{sx2-3} \cot\phi_{III} = \rho_y f_{sy-y} \tan\phi_{III} . \quad (33)$$

Equation 33 implies that the ultimate shear strength of orthogonally RC membrane elements occurs when the contribution of the longitudinal x-reinforcement to resist shear stresses is equal to that of the transverse y-reinforcement.

The ultimate shear strength may thus be expressed in terms of either x or y reinforcement as follows:

$$v_{u-III} = \rho_x \sigma_{sx2-3} \cot\phi_{III} = \rho_y f_{sy-y} \tan\phi_{III} . \quad (34)$$

Equation 31 (or 34) shows that following the yield of the y reinforcement, but prior to concrete failure, the RC membrane element can still carry additional shear stresses due to the fact that normal stresses in the x reinforcement can increase since this x reinforcement is still in the elastic range. The normal stress in the x reinforcement at the location of the second major critical cracks keeps increasing until the contribution of the x reinforcement becomes equal to that of the y reinforcement (Eq. 33). At this instant and as the load continues to increase, this condition requires an increase in the normal stresses in both reinforcements at the same time which is not possible, because the y reinforcement has already reached its yield strength. This condition leads to an increase in concrete stresses at the location of the second major critical cracks to carry the additional loads. This increase in concrete stresses eventually leads to concrete failure prior to the yield of x reinforcement for mode III (or IV).

Hence, in mode III (or IV), since the x reinforcement does not yield (i.e., $\sigma_{sx2-3} < f_{sy-x}$), the normal stresses σ_{sx2-3} are limited to those that cause splitting of concrete or those that cause concrete crushing. Concrete splitting was found to control σ_{sx2-3}, because it gives lower shear stresses. The tensile splitting strength of concrete is lost at the formation of the second major critical cracks and is temporarily compensated by the steel shear stresses (Zararis, 1996) in the x reinforcement such that:

$$f'_{sp} = \rho_x \tau_{sxy} \sin\phi_{III} \cos\phi_{III} ; \quad (35)$$

where the shear stresses in the x reinforcement at the location of the second major critical cracks can be expressed (similar to Eq. 7) as follows:

$$\tau_{sxy} \cong 0.40 \sigma_{sx2-3} \tan\phi_{III} . \quad (36)$$

Therefore, the normal stress σ_{sx2-3} is limited to the tensile splitting strength of concrete according to the following formula:

$$\rho_x \sigma_{sx2-3} = \frac{f'_{sp}}{0.40 \sin^2 \phi_{III}} . \quad (37)$$

Combining Eq. 32 and 37 leads to the following expression for the inclination of the second major critical cracks for mode III:

$$\sin^4 \phi_{III} + \frac{f'_{sp}}{0.40 \rho_y f_{sy-y}} \sin^2 \phi_{III} - \frac{f'_{sp}}{0.40 \rho_y f_{sy-y}} = 0 . \quad (38)$$

The solution of Eq. 38 for ϕ_{III} can explicitly be given as follows:

$$\sin\phi_{III} = \sqrt{\sqrt{\left(\frac{f'_{sp}}{0.80 \rho_y f_{sy-y}}\right)^2 + \frac{f'_{sp}}{0.40 \rho_y f_{sy-y}}} - \frac{f'_{sp}}{0.80 \rho_y f_{sy-y}}} . \quad (39)$$

Different formulae were suggested for the tensile splitting strength of concrete; two formulae used for high and low concrete strengths (Carrasquillo et al., 1981; Zararis, 1996) are shown in Fig. 7. The following single expression is proposed to combine the suggested formulae:

$$f'_{sp} = 0.25 \left(f'_c\right)^{0.70} ; \quad (40)$$

f'_{sp} and f'_c are in MPa.

Equation 39 and 40 can be used to obtain ϕ_{III} (the inclination of the second major critical cracks for mode III), which in turn can be used to obtain the shear strength (Eq. 34) and stress σ_{sx2-3} at the location of the second major critical cracks that can be expressed as follows:

$$\sigma_{sx2-3} = \frac{\rho_y f_{sy-y} \tan^2 \phi_{III}}{\rho_x} . \quad (41)$$

Table 2 shows that the estimated values of the shear strength of RC membrane elements that failed by mode III (or IV) predicted by Eq. 31 are slightly conservative with a mean value of 1.05 and a standard deviation of 0.11 for the ratio of experimental to predicted shear strength values.

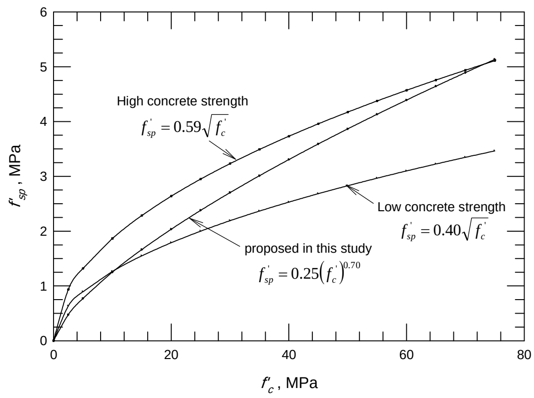

Figure (7): Proposed formula for tensile splitting strength of concrete compared to two existing formulae for low and high concrete strength

Comparison of Proposed Model with Experimental Shear Strength and Those of MCFT

Figure 8 shows that the shear strengths predicted by the proposed model for the four modes of failure are in excellent agreement with the experimental values of shear strength. Figure 9 also shows that an excellent agreement exists between the shear strength predicted by the proposed model for the four modes of failure and shear strength predicted by the MCFT (1986). Experimental results were obtained from Vecchio and Collins (1986), Zhang and Hsu (1995), Vecchio et al. (1986), Kirschner (1986) and Yamaguchi (1988).

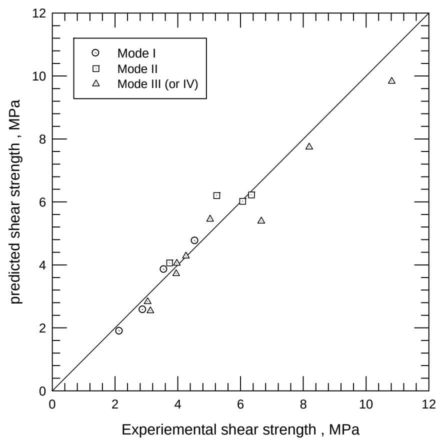

Figure (8): Comparison of predicted and experimental shear strengths for all elements that failed in the four different modes

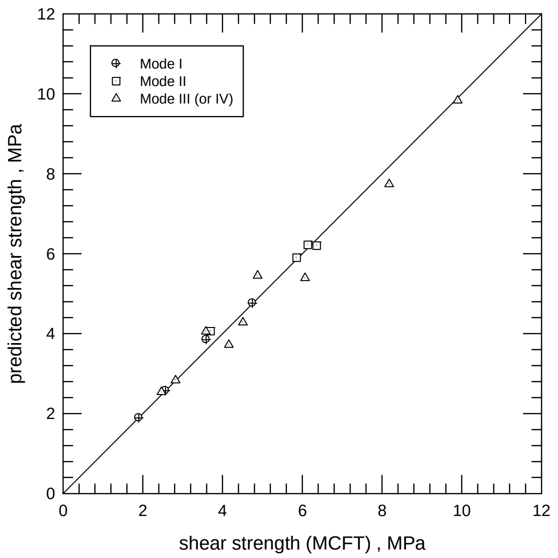

Figure (9): Comparison of predicted shear strength and those of MCFT for all elements that failed in the four different modes

SUMMARY AND CONCLUSIONS

This study is concerned with the development of a mathematical model for evaluating the ultimate shear capacity of reinforced concrete (RC) membrane elements subjected to pure in-plane shear stresses that was shown to fail in four different failure modes. The shear capacity of these membrane elements is obtained in this study by direct simple expressions without the need for any iterative procedures. Failure in these membrane elements is generally shown to be preceded by the formation of two or more major critical cracks, which are assumed to propagate in a direction normal to the principal tensile stress in concrete. The first major critical cracks are shown to open at an optimum angle that corresponds to the least shear resistance to the external cracking loads. The second major critical cracks are shown to occur at an optimum angle that corresponds to the ultimate shear failure that provides least resistance to external loading. Experimental results of tests on RC membrane elements subjected to pure in-plane shear stresses obtained from literature show that excellent agreement exists between the shear capacity predicted by the proposed model for the four modes of failure and the experimental shear capacity as well as those predicted by the MCFT. The overall mean and standard deviation for the ratio of experimental to predicted values of shear capacity, are 1.09 and 0.11, respectively.

Appendix A (Examples: Mode of Failure and Shear Strength Calculation)

Example 1:

Calculate the shear strength of specimen PV6 of Vecchio and Collins (1986) reinforced with $\rho_x =$ 0.0179; $\rho_y = 0.0179$; $f_c' =$ 29.8 MPa; $f_{sy-x} = 266$ MPa; $f_{sy-y} = 266$ MPa (Table 2).

Step 1: Calculate balanced steel ratios:

$\rho_{x-b} = 0.57\, f_c'^{0.75} \,/\, f_{sy-x} = 0.0273$ (Eq. 24)

$\rho_{y-b} = 0.57\, f_c'^{0.75} \,/\, f_{sy-y} = 0.0273$ (Eq. 25)

Step 2: Mode of failure:

$\rho_x = 0.0179 < \rho_{x-b} = 0.0273$;

$\rho_y = 0.0179 < \rho_{y-b} = 0.0273$. Therefore, both the x and y reinforcements yield; the element is thus fully under-reinforced in both x and y directions (i.e., failure mode I).

Step 3: Compute:

$v_{u-I} = \sqrt{0.0179(266)0.0179(266)} = 4.76$ MPa (Eq. 10) compared to 4.55 MPa (test); 4.75 MPa (MCFT of Vecchio and Collins, 1986), 4.74 MPa (SMCS of Rahal, 2008).

Example 2:

Calculate the shear strength of specimen PV9 of Vecchio and Collins (1) reinforced with $\rho_x = 0.0179$; $\rho_y = 0.0179$; $f_c' =$ 11.6 MPa; $f_{sy-x} = 455$ MPa; $f_{sy-y} = 455$ MPa (Table 2).

Step 1: Calculate balanced steel ratios:

$\rho_{x-b} = 0.57\, f_c'^{0.75} \,/\, f_{sy-x} = 0.0079$ (Eq. 24)

$\rho_{y-b} = 0.57\, f_c'^{0.75} \,/\, f_{sy-y} = 0.0079$ (Eq. 25)

Step 2: Mode of failure:

$\rho_x = 0.0179 > \rho_{x-b} = 0.0079$;

$\rho_y = 0.0179 > \rho_{y-b} = 0.0079$. Therefore, both the x and y reinforcements do not yield; the element is thus fully over-reinforced in both directions (i.e., failure mode II).

Step 3: Compute: $v_{u-II} = 0.60(11.6)^{0.75}$ =3.77 MPa (Eq. 27) compared to 3.74 MPa (test); 3.70 MPa (MCFT of Vecchio and Collins, 1986), 3.71 MPa (SMCS of Rahal, 2008).

Example 3: Calculate the shear strength of specimen PV20 of Vecchio and Collins (1986) reinforced with $\rho_x = 0.0179$; $\rho_y = 0.0089$; $f'_c =$ 19.6 MPa; $f_{sy-x} =$ 460 MPa; $f_{sy-y} =$ 297 MPa (Table 2).

Step 1: Calculate balanced steel ratios:

$\rho_{x-b} = 0.57(19.6)^{0.75} / 460 = 0.0115$

and $\rho_{y-b} = 0.57(19.6)^{0.75} / 297 = 0.0179$.

Step 2: Mode of failure:

$\rho_x = 0.0179 > \rho_{x-b} = 0.0115$;

$\rho_y = 0.0089 < \rho_{y-b} = 0.01788$. Therefore, the y reinforcement yields whereas the x reinforcement does not yield; the element is thus partially under-reinforced in y direction (i.e., failure mode III).

Step 3: Compute:

$f'_{sp} = 0.25(19.6)^{0.70} = 2.007$ (Eq. 38) and $\phi_{III} = 58.4^o$ (Eq. 37).

Step 4: Compute ultimate shear capacity:

$v_{u-III} = 0.0089\,(297)\tan 58.4 = 4.28$ MPa (Eq. 34) compared to 4.26 MPa (test); 4.51 MPa (MCFT of Vecchio and Collins, 1986), 4.01 MPa (SMCS of Rahal, 2008).

Step 5: Compute normal stress at location of the second major critical cracks:

$$\sigma_{sx2-3} = \frac{0.0089(297)\tan^2 58.4}{0.0179} = 390 \text{ MPa} < f_{sy-x} = 460 \text{ MPa (Eq. 41).}$$

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